SHEAR AND TORSION IN THICK-WALLED CROSS-SECTIONS (GENERAL FORMULATION)

We consider a thick-walled cross-section of area A with shear center at (y_{sc}, z_{sc}). The boundary condition of zero external shear traction is imposed: $T = 0$ on ∂A. The axes y, z are not principal, so the product of inertia $I_{yz} \neq 0$. The shear modulus is G.

NOTATION

V_y, V_z	shear resultants (forces) in y and z
T_w	warping torsional moment
T_t	Saint-Venant (primary) torsion
I_y, I_z	second moments of area about y and z
I_{yz}	product of inertia (nonzero for nonprincipal axes)
I_ω	warping constant (sectorial moment of inertia)
$\omega(y,z)$	primary warping function (from Section 1)
u (y,z)	additional axial displacement (from Section 2)
u_{const}	total axial displacement
τ_{yz}, τ_{zy}	shear stresses

1 PRIMARY WARPING $\omega(y,z)$ DUE TO PRIMARY TORSION T_t

1.1 Weak (variational) formulation

Find $\omega \in H^1(A)$ such that for all δw$\in H^1(A)$:

$$\int_A G\,\nabla\delta\text{w}\cdot\nabla\omega\,dA - \int_A G\left[(y-y_{sc})\frac{\partial(\delta\text{w})}{\partial z} - (z-z_{sc})\frac{\partial(\delta\text{w})}{\partial y}\right]dA = 0.$$

The boundary term vanishes because $T = 0$ on ∂A.

1.2 Strong formulation (Laplace equation)

The warping function ω satisfies:

$$\nabla^2\omega = 0 \quad \text{in } A \qquad \text{(Laplace equation)}$$

with natural (Neumann) boundary condition:

$$\frac{\partial\omega}{\partial n} = (y-y_{sc})\,n_z - (z-z_{sc})\,n_y \quad \text{on } \partial A$$

where $\boldsymbol{n} = (n_y, n_z)$ is the outward unit normal to ∂A.

1.3 Relation with the Saint–Venant torsion T_t

The rate of twist $\theta' = \frac{d\theta}{d\text{x}}$ is related to the Saint–Venant torsion T_t by

$$T_t = G I_T\,\theta'$$

where the torsional constant I_T (Saint–Venant torsional constant) is

$$I_T = \int_A \left[(y-y_{sc})^2 + (z-z_{sc})^2 + (y-y_{sc})\frac{\partial\omega}{\partial z} - (z-z_{sc})\frac{\partial\omega}{\partial y}\right]dA$$

2 ADDITIONAL AXIAL DISPLACEMENT $u_c(\text{x},y,z)$ DUE TO SHEAR FORCES (V_y,V_z) AND WARPING TORSIONA MOMENT T_w

2.1 Weak (variational) formulation

Find $u \in H^1(A)$ such that for all $\delta u \in H^1(A)$:

$$\int_A G\,\nabla\delta u\cdot\nabla u\,dA = \int_A \left[\alpha y + \beta z + \frac{T_w}{J_w}\right]\delta u\,dA$$

where

$$\alpha = \frac{V_y I_z - V_z I_{yz}}{I_y I_z - I_{yz}^2}, \qquad \beta = \frac{V_z I_y - V_y I_{yz}}{I_y I_z - I_{yz}^2}$$

V_y, V_z = shear resultants in y and z

T_w = warping torsional moment

I_y, I_z = second moments of area about y and z

I_{yz} = product of inertia (nonzero for nonprincipal axes)

I_ω = warping constant (sectorial moment of inertia)

$\omega(y,z)$ = primary warping function (from Section 1)

2.2 Strong formulation (Poisson equation)

The function $u(\text{x},y,z)$ satisfies:

$$-G\,\nabla^2 u = \alpha y + \beta z + \frac{T_w}{J_w}\,u \qquad \text{in } A$$

with natural boundary condition:

$$G\,\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial A$$

Important: u depends on α because α, β (from V_y, V_z) and $\frac{T_w}{T_w}$ (from T_w) depend on x.
Typically: $V_y = V_y(\text{x}), V_z = V_z(\text{x}),\ T_w = T_w(\text{x})$ so $u = u(\text{x}, y, z)$.

3 TOTAL DISPLACEMENT FIELD AND SHEAR STRAINS

3.1 Total axial displacement

At a point (x, y, z):

$$u_{total}(\text{x},y,z) = u_c(\text{x}) - y\,\vartheta_z(\text{x}) + z\,\vartheta_y(\text{x}) - \omega(y,z)\,\theta'(\text{x}) + u(\text{x},y,z)$$

v(x,y,z)=v(x)-(z-zsc)θx   w(x,y,z)=w(x)+(y-ysc)θx

$u_c(\text{x})$ = axial displacement of the centroid

$\vartheta_y(\text{x}), \vartheta_c(\text{x})$ = bending rotations (about y and z)

$\theta'(\text{x}) = \frac{d\theta}{d\text{x}}$ = rate of twist

$\omega(y,z)$ = primary warping function (from Section 1)

$u(\text{x},y,z)$ = additional axial displacement (from Section 2)

3.2 Engineering shear strains (from total displacement)

Shear strains associated with shear stresses:

$$\gamma_{\text{xy}} = \frac{\partial u_{total}}{\partial y} + \frac{\partial \text{v}}{\partial \text{x}} = \left[-(z-z_{sc}) - \frac{\partial\omega}{\partial y}\right]\theta' + \frac{\partial u}{\partial y}$$

$$\gamma_{\text{xz}} = \frac{\partial u_{total}}{\partial z} + \frac{\partial \text{w}}{\partial \text{x}} = \left[(y-y_{sc}) - \frac{\partial\omega}{\partial z}\right]\theta' + \frac{\partial u}{\partial z}$$

4 GENERAL EXPRESSIONS FOR SHEAR STRESSES

Using Hooke's law for shear: $\tau = G\,\gamma$.

4.1 Primary shear stresses (due to pure torsion T_t)

$$\tau_{\text{xy}}^{\text{prim}} = G\left[-(z-z_{sc}) - \frac{\partial\omega}{\partial y}\right]\theta', \qquad \tau_{\text{xz}}^{\text{prim}} = G\left[(y-y_{sc}) - \frac{\partial\omega}{\partial z}\right]\theta'$$

4.2 Secondary shear stresses (due to shear forces and warping torsion)

$$\tau_{\text{xy}}^{\text{sec}} = G\,\frac{\partial u}{\partial y}, \qquad \tau_{\text{xz}}^{\text{sec}} = G\,\frac{\partial u}{\partial z}$$

4.3 Total shear stresses

$$\tau_{\text{xy}} = G\left[-(z-z_{sc}) - \frac{\partial\omega}{\partial y}\right]\theta' + G\,\frac{\partial u}{\partial y}$$

$$\tau_{\text{xz}} = G\left[(y-y_{sc}) - \frac{\partial\omega}{\partial z}\right]\theta' + G\,\frac{\partial u}{\partial z}$$

Resultant shear stress at a point:

$$\tau = \sqrt{\tau_{\text{xy}}^2 + \tau_{\text{xz}}^2}$$

5 REMARKS ON NONPRINCIPAL AXES

- Because $I_{yz} \neq 0$, the shear-force terms in Section 2 are coupled: the coefficients α and β involve I_{yz}.
- The shear center (y_{sc}, z_{sc}) is a property of the cross-section and is independent of the axis system.
- The warping function ω is obtained by solving the Laplace equation with the Neumann condition in Section 1.
- Once ω is known, the Poisson equation in Section 2 is solved for u with the natural boundary condition.

6 SUMMARY OF THE SOLUTION PROCEDURE

① **Solve for $\omega(y,z)$:** Solve $\nabla^2\omega = 0$ in A with $\frac{\partial\omega}{\partial n} = (y-y_{sc})\,n_z - (z-z_{sc})\,n_y$ on ∂A.

② **Compute I_T:** Evaluate $I_T = \int_A \left[(y-y_{sc})^2 + (z-z_{sc})^2 + (y-y_{sc})\frac{\partial\omega}{\partial z} - (z-z_{sc})\frac{\partial\omega}{\partial y}\right]dA$.

③ **Solve for $u(z,y,z)$:** Solve $-G\,\nabla^2 u = \alpha y + \beta z + \frac{T_w}{J_w}$;u in A with $G\,\frac{\partial u}{\partial n} = 0$ on ∂A.

④ **Compute shear stresses:** Use the total stress expressions in Section 4
τ_{xy} τ_{xz} (or keep in terms of θ').

⑤ **Resultant shear stress:** $\tau = \sqrt{\tau_{\text{xy}}^2 + \tau_{\text{xz}}^2}$ at any point.

This framework is fully general for solid or thick-walled sections and provides the governing equations for shear, torsion, and warping in combined shear, torsion, and warping torsion.