Finite Element Formulation for Lateral–Torsional Buckling

2-Node Beam Element — Umansky Hypothesis — Weak Form

Linearly Varying Bending Moment – Wagner Term Without 1/2 Factor

1 Degrees of Freedom and Order

Each node has four degrees of freedom.

The global element order is:

$$\mathbf{d}^e = \left[\, v_1 \quad v_1' \quad \theta_1 \quad \psi_1 \quad v_2 \quad v_2' \quad \theta_2 \quad \psi_2 \,\right]^T$$

- v: lateral displacement
- v': rotation due to bending
- θ: torsional rotation
- ψ: warping intensity

2 Umansky Hypothesis

$$k = 1 - \frac{J}{I_p}$$

- J: Saint–Venant torsional constant
- I_p: polar moment of inertia

For closed sections $k \approx 0 \rightarrow$ torsion is dominated by Saint–Venant.

Warping ψ contributes only to elastic stiffness, not to geometric stiffness.

3 Weak Form

3.1 Elastic contribution (torsion + warping)

$$\delta W_{int} = \int_0^L \left[EI_\omega\, \phi'\, \delta\phi' + \frac{kGJ}{1-k}\, (\theta' - \phi)\, (\delta\theta' - \delta\phi) + GJ\, \theta'\, \delta\theta' \right] dx$$

3.2 Geometric contribution (pre-stress $M_y(x)$)

$$\delta W_G = \int_0^L M_y(x)\, (v''\, \delta\theta + \delta v''\, \theta)\, dx + \int_0^L M_y(x)\, \beta_z\, \theta'\, \delta\theta'\, dx$$

Note: The Wagner term does NOT include the factor 1/2.

The bending moment varies linearly:

$$M_y(x) = M_{y1} \left(1 - \frac{x}{L}\right) + M_{y2} \left(\frac{x}{L}\right)$$

4 Interpolation Functions

Dimensionless coordinate $\xi = x/L$.

4.1 Lateral displacement (cubic Hermite)

$$v(\xi) = \mathbf{N}_v(\xi)\, \mathbf{d}_v, \qquad \mathbf{d}_v = \left[\, v_1 \quad v_1' \quad v_2 \quad v_2' \,\right]^T$$

Curvature: $v''(\xi) = \mathbf{B}_v(\xi)\, \mathbf{d}_v$ with

$$\begin{cases} B_{v1}(\xi) = \dfrac{1}{L^2}(-6 + 12\xi), & B_{v2}(\xi) = \dfrac{1}{L}(-4 + 6\xi), \\ B_{v3}(\xi) = \dfrac{1}{L^2}(6 - 12\xi), & B_{v4}(\xi) = \dfrac{1}{L}(-2 + 6\xi). \end{cases}$$

4.2 Torsion (linear)

$$\theta(\xi) = (1-\xi)\theta_1 + \xi\theta_2, \qquad \theta' = \frac{\theta_2 - \theta_1}{L}$$

4.3 Warping (linear)

$$\phi(\xi) = (1-\xi)\psi_1 + \xi\psi_2$$

5 Elastic Stiffness Matrix K_e

K_e is block diagonal in the global order.

5.1 Bending block (DOFs 1,2,5,6)

$$K_e^{vv} = \frac{EI_z}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

5.2 Torsion–warping block (DOFs 3,4,7,8)

Define $\alpha = \dfrac{GJ}{1-k}, \quad \gamma = \dfrac{kGJ}{1-k}$.

$$K_e^{\theta\psi} = \begin{bmatrix} \dfrac{\alpha}{L} & \gamma & -\dfrac{\alpha}{L} & 0 \\ \gamma & \dfrac{\gamma L}{3} + \dfrac{EI_\omega}{L} & 0 & \dfrac{\gamma L}{6} - \dfrac{EI_\omega}{L} \\ -\dfrac{\alpha}{L} & 0 & \dfrac{\alpha}{L} & -\gamma \\ 0 & \dfrac{\gamma L}{6} - \dfrac{EI_\omega}{L} & -\gamma & \dfrac{\gamma L}{3} + \dfrac{EI_\omega}{L} \end{bmatrix}$$

6 Geometric Stiffness Matrix K_G (without 1/2)

6.1 Bending–torsion coupling

$$K_G^{(v\theta)} = \int_0^L \mathbf{B}_v^T\, M_y(x)\, \mathbf{N}_\theta\, dx$$

Introduce the auxiliary matrices (4 rows, 2 columns):

$$\mathbf{C}_0 = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \\ -1 & 0 \\ \frac{1}{L} & -\frac{1}{L} \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C}_1 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{L} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

Then:

$$K_G^{(v\theta)} = M_{y1}\, (\mathbf{C}_0 - \mathbf{C}_1) + M_{y2}\, \mathbf{C}_1$$

Explicitly:

$$K_G^{(v\theta)} = M_{y1} \begin{bmatrix} -\frac{1}{L} & 0 \\ -\frac{5}{6} & -\frac{1}{6} \\ \frac{1}{L} & 0 \\ -\frac{1}{6} & \frac{1}{6} \end{bmatrix} + M_{y2} \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{L} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

6.2 Wagner term (pure torsion)

$$K_G^{(\theta\theta)} = \frac{\beta_z (M_{y1} + M_{y2})}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

6.3 Global assembly (8 × 8)

Row and column order:

$$[\, v_1,\ v_1',\ \theta_1,\ \psi_1,\ v_2,\ v_2',\ \theta_2,\ \psi_2 \,]$$

$$K_G = \begin{bmatrix} 0 & 0 & K_{13} & 0 & 0 & 0 & K_{17} & 0 \\ 0 & 0 & K_{23} & 0 & 0 & 0 & K_{27} & 0 \\ K_{31} & K_{32} & K_{33} & 0 & K_{35} & K_{36} & K_{37} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{53} & 0 & 0 & 0 & K_{57} & 0 \\ 0 & 0 & K_{63} & 0 & 0 & 0 & K_{67} & 0 \\ K_{71} & K_{72} & K_{73} & 0 & K_{75} & K_{76} & K_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where:

$$K_{13} = -\frac{M_{y1}}{L} \qquad K_{17} = \frac{M_{y2}}{L}$$

$$K_{23} = -\frac{5}{6} M_{y1} - \frac{1}{6} M_{y2} \qquad K_{27} = -\frac{1}{6} M_{y1} + \frac{1}{6} M_{y2}$$

$$K_{53} = \frac{M_{y1}}{L} \qquad K_{57} = -\frac{M_{y2}}{L}$$

$$K_{63} = -\frac{1}{6} M_{y1} + \frac{1}{6} M_{y2} \qquad K_{67} = \frac{1}{6} M_{y1} + \frac{5}{6} M_{y2}$$

$$K_{33} = \frac{\beta_z (M_{y1} + M_{y2})}{2L} \qquad K_{37} = \frac{\beta_z (M_{y1} + M_{y2})}{2L}$$

$$K_{73} = K_{37} \qquad K_{77} = K_{33}$$

and the symmetric terms:

$$K_{31} = K_{13}, \quad K_{32} = K_{23}, \quad K_{35} = K_{53}, \quad K_{36} = K_{63},$$
$$K_{71} = K_{17}, \quad K_{72} = K_{27}, \quad K_{75} = K_{57}, \quad K_{76} = K_{67}.$$

7 Wagner Coefficient β_z

$$\beta_z = \frac{1}{I_y} \int_A z\, (y^2 + z^2)\, dA - 2z_0$$

- (y,z): centroidal coordinates of the section
- $I_y = \int_A z^2\, dA$
- z_0: z-coordinate of the shear center

8 Eigenvalue Problem

$$(\mathbf{K}_e + \lambda \mathbf{K}_G)\, \mathbf{d} = 0$$

The parameter λ scales the moment distribution $\{M_{y1}, M_{y2}\}$. The critical moments are λM_{y1} at node 1 and λM_{y2} at node 2.

Observations:

- Rows and columns corresponding to ψ_1 (index 4) and ψ_2 (index 8) are zero.
- The matrix is symmetric by construction.
- For constant moment ($M_{y1} = M_{y2} = M$), the Wagner block reduces to
  $$\frac{\beta_z M}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$
  and the bending–torsion coupling simplifies to $M\,\mathbf{C}_0$.

9 Interpretative Summary

Block	Origin	Physical Effect
K_e^{vv}	Elastic bending	Lateral stiffness
$K_e^{\theta\psi}$	Torsion + warping + coupling	Torsional and warping stiffness
$K_G^{(v\theta)}$	$M_y(x)\,(v''\,\delta\theta + \delta v''\,\theta)$	Bending–torsion coupling
$K_G^{(\theta\theta)}$	$M_y(x)\,\beta_z\,\theta'\,\delta\theta'$	Wagner effect (without 1/2)

Key Conclusion:

The warping degrees of freedom ψ participate only in the elastic stiffness; the geometric stiffness matrix does not activate them, in accordance with the Umansky hypothesis for closed sections. The presented formulation is valid for a linearly varying bending moment along the element.