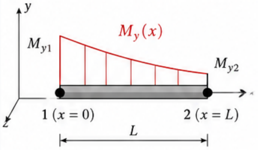
Geometric Stiffness Matrix of a Beam–Column Element

Including Torsion–Warping (Exact Shape Functions) and Lateral Bending (Hermite Cubics)

1) Virtual Work of the Geometric Contribution

Considering the contribution of the (linearly varying) bending moment $M_y(x)$, the virtual work is

$$\delta W_G = \int_0^L M_y(x)\left(v''\delta\theta + \delta v''\theta\right)\mathrm{d}x + \int_0^L M_y(x)\,\beta_z\theta'\delta\theta'\,\mathrm{d}x$$



2) Approximations in the Element ($x \in [0, L],\ \xi = x/L$)

Lateral displacement v (Hermite cubic functions)

$$v(x) = \mathbf{N}_v(\xi)\,v_e, \qquad v_e = [v_1,\ v_1',\ v_2,\ v_2']^T$$

Shape function	$\mathbf{N}_{vi}(\xi)$	$\mathrm{d}^2\mathbf{N}_{vi}/\mathrm{d}x^2$
$N_{v1}(\xi)$	$= 1 - 3\xi^2 + 2\xi^3$	$\dfrac{-6+12\xi}{L^2}$
$N_{v2}(\xi)$	$= L(\xi - 2\xi^2 + \xi^3)$	$\dfrac{-4+6\xi}{L}$
$N_{v3}(\xi)$	$= 3\xi^2 - 2\xi^3$	$\dfrac{6-12\xi}{L^2}$
$N_{v4}(\xi)$	$= L(-\xi^2 + \xi^3)$	$\dfrac{-2+6\xi}{L}$

$$v''(x) = \frac{1}{L^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\mathbf{N}_v(\xi)\,v_e$$

Total twist $\theta(x)$ (torsion–warping, exact functions)

$$\theta(x) = \mathbf{N}_\theta(x)\,\boldsymbol{\theta}_e, \qquad \boldsymbol{\theta}_e = [\theta_1,\ \varphi_1,\ \theta_2,\ \varphi_2]^T$$

The shape functions are the exact ones obtained previously:

$\mathbf{N}_{\theta 1}(x) = N_{\theta\theta_0}(x)$

$\mathbf{N}_{\theta 2}(x) = N_{\theta\varphi_0}(x)$

$\mathbf{N}_{\theta 3}(x) = N_{\theta\theta_L}(x)$

$\mathbf{N}_{\theta 4}(x) = N_{\theta\varphi_L}(x)$

Parameters

$\alpha = \sqrt{\kappa GJ/EI_\omega}, \quad \mu = \alpha/\kappa$

$\lambda = \alpha, \quad s = \sinh(\alpha L),$

$c = \cosh(\alpha L)$

$s_x = \sinh(\alpha x), \quad c_x = \cosh(\alpha x)$

$\Delta = 2(1-c) + \mu\alpha L s$

$$\mathbf{N}_\theta'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}_\theta(x)$$

Linearly varying bending moment

$$M_y(x) = M_{y1}\left(1 - \frac{x}{L}\right) + M_{y2}\frac{x}{L} = M_{y1}N_1^M(\xi) + M_{y2}N_2^M(\xi),$$
$$N_1^M(\xi) = 1 - \xi, \qquad N_2^M(\xi) = \xi$$

3) Geometric Stiffness Matrix of the Element

The elemental geometric stiffness matrix (symmetric, 8×8) is

$$\mathbf{K}_G = \begin{bmatrix} \mathbf{0}_{4\times 4} & \mathbf{K}_{v\theta} \\ \mathbf{K}_{v\theta}^T & \mathbf{K}_{\theta\theta} \end{bmatrix}$$

Element DOF vector

$$\mathbf{d}_e = [v_1,\ v_1',\ v_2,\ v_2',\ \theta_1,\ \varphi_1,\ \theta_2,\ \varphi_2]^T$$

Coupling (4×4)

$$\mathbf{K}_{v\theta} = \int_0^L M_y(x)\left(\mathbf{N}_v''(x)\right)^T\mathbf{N}_\theta(x)\,\mathrm{d}x$$

Torsion–warping block (4×4)

$$\mathbf{K}_{\theta\theta} = \beta_z\int_0^L M_y(x)\left(\mathbf{N}_\theta'(x)\right)^T\mathbf{N}_\theta'(x)\,\mathrm{d}x$$

- $\mathbf{N}_v''(x) = \frac{1}{L^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\mathbf{N}_v(\xi)$ with $\xi = x/L$.
- $\mathbf{N}_\theta(x)$: exact torsion–warping shape functions (functions of $x, \alpha, \mu, s, c, s_x, c_x, \Delta$).
- $\mathbf{N}_\theta'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}_\theta(x)$.
- $M_y(x) = M_{y1}(1 - x/L) + M_{y2}(x/L)$ (linear).

Explicit expressions of $\mathbf{N}_{\theta i}(x)$
(from previous derivation)

$N_{\theta\theta_0}(x),\ N_{\theta\varphi_0}(x),\ N_{\theta\theta_L}(x),\ N_{\theta\varphi_L}(x)$

in terms of $c, s, c_x, s_x, \alpha, \mu, \Delta$.

4) Remarks

- The integrals can be evaluated analytically (symbolic computation), yielding closed-form expressions involving polynomials and hyperbolic functions of αL.
- $\mathbf{K}_G$ provides the exact geometric contribution for a beam element with linear bending moment $M_y(x)$, coupling lateral bending (v) with torsion–warping (θ, φ).
- The matrix is symmetric and consistent with the virtual work statement.

Dimensions

$\mathbf{K}_G \in \mathbb{R}^{8\times 8}$

(4 DOF for v) +

(4 DOF for θ, φ)