Exact Shape Functions for Torsion (θ_x) and Warping (φ)

The exact shape functions for the kinematic variables θ_x (torsion) and φ (warping) in terms of their nodal values at $x = 0$ and $x = L$ are obtained by solving the homogeneous system $\boldsymbol{E}' = \boldsymbol{W}\boldsymbol{E}$. With the parameters:

$$\lambda = \sqrt{\frac{\kappa GJ}{EI_\omega}}, \quad \mu = \frac{\lambda}{\kappa}, \quad s = \sinh(\lambda L), \quad c = \cosh(\lambda L), \quad s_x = \sinh(\lambda x), \quad c_x = \cosh(\lambda x)$$

and the determinant

$$\Delta = 2(1-c) + \mu\lambda L\, s = 2(1-c) + \frac{\lambda^2 L}{\kappa} s$$

the shape functions are:

$N_{\theta\theta_0}(x)$	$= 1 - \dfrac{(c-1)(c_x-1) - s\,s_x + \mu s\lambda x}{\Delta}$
$N_{\theta\theta_L}(x)$	$= \dfrac{(c-1)(c_x-1) - s\,s_x + \mu s\lambda x}{\Delta}$
$N_{\theta\varphi_0}(x)$	$= x - \dfrac{L[(c-1)(c_x-1) - s\,s_x + \mu s\lambda x] + [-(s-\mu\lambda L)(c_x-1) + (c-1)(s_x-\mu\lambda x)]}{\Delta}$
$N_{\theta\varphi_L}(x)$	$= \dfrac{-(s-\mu\lambda L)(c_x-1) + (c-1)(s_x-\mu\lambda x)}{\Delta}$
$N_{\varphi\theta_0}(x)$	$= -\dfrac{\mu[(c-1)s_x - s(c_x-1)]}{\Delta}$
$N_{\varphi\theta_L}(x)$	$= \dfrac{\mu[(c-1)s_x - s(c_x-1)]}{\Delta}$
$N_{\varphi\varphi_0}(x)$	$= 1 - \dfrac{\mu L[(c-1)s_x - s(c_x-1)] + \mu[-(s-\mu\lambda L)s_x + (c-1)(c_x-1)]}{\Delta}$
$N_{\varphi\varphi_L}(x)$	$= \dfrac{\mu[-(s-\mu\lambda L)s_x + (c-1)(c_x-1)]}{\Delta}$

Approximation:

$$\theta_x(x) = N_{\theta\theta_0}(x)\,\theta_x(0) + N_{\theta\varphi_0}(x)\,\varphi(0) + N_{\theta\theta_L}(x)\,\theta_x(L) + N_{\theta\varphi_L}(x)\,\varphi(L)$$

$$\varphi(x) = N_{\varphi\theta_0}(x)\,\theta_x(0) + N_{\varphi\varphi_0}(x)\,\varphi(0) + N_{\varphi\theta_L}(x)\,\theta_x(L) + N_{\varphi\varphi_L}(x)\,\varphi(L)$$

These functions satisfy exactly the boundary conditions at the nodes and reproduce exactly the homogeneous solution of the system of ODEs.