GEOMETRIC STIFFNESS MATRIX (8×8) AND BUCKLING LOAD P_{cr}

FLEXURAL–TORSIONAL BUCKLING OF THIN-WALLED OPEN SECTIONS

1 VARIATIONAL FORMULATION (WEAK FORM)

Elastic contribution

$$\delta W_{el} = \int_0^L \left[EI_z\, v''\, \delta v'' + EI_\omega\, \phi'\, \delta\phi' + \frac{kGJ}{1-k}(\theta' - \phi)(\delta\theta' - \delta\phi) + GJ\,\theta'\,\delta\theta' \right] dx$$

Geometric contribution (constant axial force $N > 0$, tension)

$$\delta W_{geom} = \int_0^L N \left[v'\,\delta v' + r_0^2\,\theta'\,\delta\theta' + z_{sc}\,(v'\,\delta\theta' + \theta'\,\delta v') \right] dx$$

Equilibrium (principle of virtual work)

$$\delta W_{el} + \delta W_{geom} = 0$$

2 STRONG FORM (0, L)

(v): $$EI_z\, v^{(4)} - N\, v'' - N\, z_{sc}\,\theta'' = 0$$

(θ): $$\left(\frac{kGJ}{1-k} + GJ + N r_0^2 \right) \theta'' - \frac{kGJ}{1-k}\,\phi' + N z_{sc}\, v'' = 0$$

(ϕ): $$EI_\omega\,\phi'' + \frac{kGJ}{1-k}\,(\theta' - \phi) = 0$$

Notation: $v(x)$ lateral displacement; $\theta(x)$ twist; $\phi(x)$ warping function

E: Young's modulus, G: shear modulus, J: St. Venant torsion constant

I_z: second moment of area about z; I_ω: warping constant

r_0: polar radius of shear center; z_{sc}: shear center offset (about y)

3 GEOMETRIC STIFFNESS MATRIX K_g (8×8)

Degrees of freedom (node 1, then node 2):

$$\mathbf{q}^T = [v_1,\ v_1',\ \theta_1,\ \phi_1,\ v_2,\ v_2',\ \theta_2,\ \phi_2]$$

- v: cubic Hermite interpolation (4 DOF)
- θ: linear interpolation (2 DOF)
- ϕ: linear interpolation (2 DOF) – no geometric contribution

$$K_g = \begin{bmatrix} K_{vv} & K_{v\theta} & 0 \\ K_{\theta v} & K_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- K_{vv} is 4×4
- $K_{v\theta}$ is 4×2
- $K_{\theta\theta}$ is 2×2
- Zero blocks have compatible dimensions

Flexural block K_{vv} (DOFs 1, 2, 5, 6)

$$K_{vv} = \frac{N}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

Coupling block $K_{v\theta}$ (rows 1, 2, 5, 6; cols 3, 7)

$$K_{v\theta} = \frac{N z_{sc}}{L} \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$K_{\theta v} = K_{v\theta}^T$$

Torsional block $K_{\theta\theta}$ (DOFs 3, 7)

$$K_{\theta\theta} = \frac{N r_0^2}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Warping degrees of freedom ϕ_1 and ϕ_2 (positions 4 and 8) do not contribute to geometric stiffness; therefore, rows and columns 4 and 8 of K_g are zero. The matrix K_g is symmetric, of size 8×8, and sparse.

4 BUCKLING LOAD P_{cr} (SIMPLY SUPPORTED: BENDING FREE, TORSION RESTRAINED, WARPING FREE)

Assumed mode shapes

$$v(x) = V \sin\frac{\pi x}{L}, \qquad \theta(x) = \Theta \sin\frac{\pi x}{L}, \qquad \phi(x) = \Phi \cos\frac{\pi x}{L}$$

Substitution in the strong form and elimination of Φ lead to a homogeneous system in V, Θ:

$$A(P) \begin{bmatrix} V \\ \Theta \end{bmatrix} = 0$$

Nontrivial solution $\Rightarrow$ $\det[A(P)] = 0$ $\Rightarrow$ general buckling equation $(P = P_{cr})$:

$$\left(EI_z \frac{\pi^4}{L^4} + P \frac{\pi^2}{L^2} \right) \left[\left(\frac{kGJ}{1-k} + GJ + P r_0^2 \right) \frac{\pi^2}{L^2} + \frac{\frac{kGJ}{1-k}}{\frac{\pi^2}{L^2} EI_\omega + \frac{kGJ}{1-k}} \right] - \left(P\, z_{sc} \frac{\pi^2}{L^2} \right)^2 = 0$$

The smallest positive root of this equation is th critical flexural–torsional buckling load P_{cr}.

5 PARTICULAR CASES

(a) Saint-Venant theory ($k = 0$)

Since $\frac{kGJ}{1-k} = 0$, the general equation reduces to:

$$\left(EI_z \frac{\pi^2}{L^2} + P \right) \left(GJ + P r_0^2 \right) - P^2 z_{sc}^2 = 0$$

(b) Vlasov theory ($k \to 1$, i.e., $\phi = \theta'$)

Taking the limit $k \to 1$ in the general equation gives:

$$\left(EI_z \frac{\pi^4}{L^4} + P \frac{\pi^2}{L^2} \right) \left(EI_\omega \frac{\pi^4}{L^4} + (GJ + P r_0^2) \frac{\pi^2}{L^2} \right) - \left(P\, z_{sc} \frac{\pi^2}{L^2} \right)^2 = 0$$

Both expressions are obtained directly from the general equation.

6 EXPLICIT STRUCTURE OF K_g (8×8)

Index order: 1: v_1, 2: v_1', 3: θ_1, 4: ϕ_1, 5: v_2, 6: v_2', 7: θ_2, 8: ϕ_2

	1	2	3	4	5	6	7	8
1	K_{vv11}	K_{vv12}	$K_{v\theta13}$	0	K_{vv15}	K_{vv16}	$K_{v\theta17}$	0
2	K_{vv21}	K_{vv22}	$K_{v\theta23}$	0	K_{vv25}	K_{vv26}	$K_{v\theta27}$	0
3	$K_{\theta v31}$	$K_{\theta v32}$	$K_{\theta\theta33}$	0	$K_{\theta v35}$	$K_{\theta v36}$	$K_{\theta\theta37}$	0
4	0	0	0	0	0	0	0	0
5	K_{vv51}	K_{vv52}	$K_{v\theta53}$	0	K_{vv55}	K_{vv56}	$K_{v\theta57}$	0
6	K_{vv61}	K_{vv62}	$K_{v\theta63}$	0	K_{vv65}	K_{vv66}	$K_{v\theta67}$	0
7	$K_{\theta v71}$	$K_{\theta v72}$	$K_{\theta\theta73}$	0	$K_{\theta v75}$	$K_{\theta v76}$	$K_{\theta\theta77}$	0
8	0	0	0	0	0	0	0	0

Only the blocks shown above are nonzero: K_{vv}, $K_{v\theta}$, $K_{\theta\theta}$ and their transposes. Rows and columns 4 and 8 (warping DOFs) are zero.

USE IN BUCKLING ANALYSIS

Assemble elastic stiffness matrix K_e and geometric stiffness matrix K_g.

Solve the generalized eigenvalue problem: $(K_e - P K_g)\, q = 0$

The smallest positive eigenvalue P_{cr} is the critical buckling load.