

2. WEAK FORM (ELASTIC CONTRIBUTION)

With the dimensionless parameter k ($0 \leq k < 1$):

$$\delta W_{\text{int}} = \int_0^L \left[EI_{\omega} \phi' \delta \phi' + \frac{kGJ}{1-k} (\theta' - \phi) (\delta \theta' - \delta \phi) + GJ \theta' \delta \theta' \right] dx$$

where E = Young's modulus, G = shear modulus,
 J = St. Venant torsion constant, I_{ω} = warping constant.

3. RELATION BETWEEN k AND UMANSKY CONSTANT J_d

Define:

$$GJ_d = \frac{k}{1-k} GJ \quad \Rightarrow$$

$$J_d = \frac{k}{1-k} J$$

$$k = \frac{J_d}{J + J_d}$$

Substituting, the weak form becomes:

$$\delta W_{\text{int}} = \int_0^L \left[EI_{\omega} \phi' \delta \phi' + GJ_d (\theta' - \phi) (\delta \theta' - \delta \phi) + GJ \theta' \delta \theta' \right] dx$$

4. EQUATIONS OF EQUILIBRIUM (EULER-LAGRANGE)

• Variation $\delta \theta$:

$$\frac{d}{dx} \left[GJ \theta' + GJ_d (\theta' - \phi) \right] = 0$$

The total torsional moment is constant
 (no distributed torsional loads):

$$M_T = GJ \theta' + GJ_d (\theta' - \phi) = \text{constant}$$

• Variation $\delta \phi$:

$$\frac{d}{dx} (EI_{\omega} \phi') + GJ_d (\theta' - \phi) = 0$$

Meaning of terms

M_T : total torsional moment

$GJ \theta'$: St. Venant torsion

$T_w = GJ_d (\theta' - \phi)$:
 warping (bitorque) torque

$EI_{\omega} \phi'$: warping moment

5. ENERGETIC DEFINITION OF J_d (ENERGY EQUIVALENCE)

Define the warping (bitorque) torque:

$$T_w = GJ_d (\theta' - \phi)$$

Strain energy per unit length associated
 with warping shear in the model:

$$u_{\text{model}} = \frac{1}{2} \frac{T_w^2}{GJ_d}$$

The real warping shear stress field τ_{warp} in the
 section produces the strain energy:

$$u_{\text{real}} = \frac{1}{2} \int_A \frac{\tau_{\text{warp}}^2}{G} dA$$

Energy equivalence principle:

$$\frac{T_w^2}{GJ_d} = \int_A \frac{\tau_{\text{warp}}^2}{G} dA \quad \Rightarrow$$

$$J_d = \frac{T_w^2}{\int_A \tau_{\text{warp}}^2 dA}$$

Analogy with Timoshenko shear area

$$\frac{1}{A_s} = \frac{1}{V^2} \int_A \tau_{\text{shear}}^2 dA$$

Same concept:
 equate strain energy of the
 real stress field to that of
 the simplified model.