1 Total strain energy (density per unit length)

$$U = \frac{1}{2}EA\left(\frac{du}{dx}\right)^2 + \frac{1}{2}EI_y\left(\frac{d\theta_y}{dx}\right)^2 + \frac{1}{2}EI_z\left(\frac{d\theta_z}{dx}\right)^2$$
$$+ EI_{yz}\frac{d\theta_y}{dx}\frac{d\theta_z}{dx} + \frac{1}{2}EI_\omega\left(\frac{d\varphi}{dx}\right)^2$$
$$+ \frac{1}{2}Gf_{yy}\gamma_y^2 + \frac{1}{2}Gf_{zz}\gamma_z^2 + \frac{1}{2}Gf_{tt}\kappa_t^2 + \frac{1}{2}Gf_{ww}\kappa_w^2$$
$$+ \frac{1}{2}Gf_{yz}\gamma_y\gamma_z + \frac{1}{2}Gf_{yw}\gamma_y\kappa_w + \frac{1}{2}Gf_{zw}\gamma_z\kappa_w$$
$$+ \frac{1}{2}Gf_{tw}\kappa_t\kappa_w,$$

Generalized strains (definitions)

$$\gamma_y = \frac{dv}{dx} - \theta_z,$$
$$\gamma_z = \frac{dw}{dx} - \theta_y,$$
$$\kappa_t = \frac{d\theta_x}{dx},$$
$$\kappa_w = \frac{d\theta_x}{dx} - \varphi.$$

E	Young's modulus
G	Shear modulus
A	Cross-sectional area
I_y, I_z	Second moments of area
I_{yz}	Product of inertia
I_ω	Warping constant
f_{**}	Shear/warping flexibility factors
u, v, w	Axial and transverse displacements
$\theta_x, \theta_y, \theta_z$	Section rotations about x, y, z
φ	Rate of twist (warping rotation)
$x \in [0, L]$	Beam axis coordinate

2 Weak formulation (principle of virtual work)

First variation of internal energy: $\delta U = \int_0^L (\delta U_{\text{class}} + \delta U_{\text{shear}})\,dx = \delta W_{\text{ext}},$

where δW_{ext} includes the virtual work of external forces and moments.

Variations

Classical (axial + bending + warping):

$$\delta U_{\text{class}} = EA\,u'\,\delta u' + EI_y\,\theta_y'\,\delta\theta_y' + EI_z\,\theta_z'\,\delta\theta_z'$$
$$- EI_{yz}\left(\theta_y'\,\delta\theta_z' + \theta_z'\,\delta\theta_y'\right) + EI_\omega\,\varphi'\,\delta\varphi',$$

Shear and warping-related:

$$\delta U_{\text{shear}} = Gf_{yy}\,\gamma_y\,\delta\gamma_y + Gf_{zz}\,\gamma_z\,\delta\gamma_z + Gf_{tt}\,\kappa_t\,\delta\kappa_t + Gf_{ww}\,\kappa_w\,\delta\kappa_w$$
$$+ \frac{1}{2}Gf_{yz}\left(\gamma_z\,\delta\gamma_y + \gamma_y\,\delta\gamma_z\right)$$
$$+ \frac{1}{2}Gf_{yw}\left(\kappa_w\,\delta\gamma_y + \gamma_y\,\delta\kappa_w\right)$$
$$+ \frac{1}{2}Gf_{zw}\left(\kappa_w\,\delta\gamma_z + \gamma_z\,\delta\kappa_w\right)$$
$$+ \frac{1}{2}Gf_{tw}\left(\kappa_w\,\delta\kappa_t + \kappa_t\,\delta\kappa_w\right).$$

Variations of the generalized strains (in terms of virtual fields)

$$\delta\gamma_y = \delta v' - \delta\theta_z, \qquad \delta\gamma_z = \delta w' - \delta\theta_y, \qquad \delta\kappa_t = \delta\theta_x', \qquad \delta\kappa_w = \delta\theta_x' - \delta\varphi.$$

Substituting the above relations and integrating by parts yields the explicit weak form, which is the basis for finite element discretization.

3 Strong formulation (equilibrium equations and boundary conditions)

Generalized stresses (resultants)

$$N = EA\frac{du}{dx},$$
$$M_y = EI_y\frac{d\theta_y}{dx} - EI_{yz}\frac{d\theta_z}{dx},$$
$$M_z = EI_z\frac{d\theta_z}{dx} - EI_{yz}\frac{d\theta_y}{dx},$$
$$B = EI_\omega\frac{d\varphi}{dx},$$
$$Q_y = Gf_{yy}\,\gamma_y + \frac{1}{2}Gf_{yz}\,\gamma_z + \frac{1}{2}Gf_{yw}\,\kappa_w,$$
$$Q_z = Gf_{zz}\,\gamma_z + \frac{1}{2}Gf_{yz}\,\gamma_y + \frac{1}{2}Gf_{zw}\,\kappa_w,$$
$$T_t = Gf_{tt}\,\kappa_t + \frac{1}{2}Gf_{tw}\,\kappa_w,$$
$$T_w = Gf_{ww}\,\kappa_w + \frac{1}{2}Gf_{yw}\,\gamma_y + \frac{1}{2}Gf_{zw}\,\gamma_z + \frac{1}{2}Gf_{tw}\,\kappa_t.$$

Equilibrium equations in the domain $(0, L)$
(with distributed loads f_u, f_v, f_w and distributed moments m_x, m_y, m_z, m_φ)

$$\frac{dN}{dx} + f_u = 0,$$
$$\frac{dQ_y}{dx} + f_v = 0,$$
$$\frac{dQ_z}{dx} + f_w = 0,$$
$$\frac{d}{dx}(T_t + T_w) + m_x = 0,$$
$$\frac{dM_y}{dx} + Q_z + m_y = 0,$$
$$\frac{dM_z}{dx} + Q_y + m_z = 0,$$
$$\frac{dB}{dx} + T_w + m_\varphi = 0.$$

Boundary conditions (at $x = 0$ and $x = L$)

For each kinematic variable, one of the two quantities in the pair must be prescribed.

Kinematic variable	Conjugate generalized stress (resultant)
u	N
v	Q_y
w	Q_z
θ_x	$T_t + T_w$
θ_y	M_y
θ_z	M_z
φ	B

Notation: $'$ denotes differentiation with respect to x (e.g., $u' = \frac{du}{dx}$, $\theta_y' = \frac{d\theta_y}{dx}$, etc.).
All cross-sectional properties (A, I_y, I_z, I_{yz}, I_ω, f_{**}) are assumed constant along the beam.