7-DOF BEAM FORMULATION WITH REPLACED GEOMETRIC COUPLINGS

The original geometric terms $-2\mathbf{M}_z\,\theta_y'\,\theta_x + 2\mathbf{M}_y\,\theta_z'\,\theta_x$ are replaced by $+\,2\mathbf{M}_y\,v'\,\theta_x' + 2\mathbf{M}_z\,w'\,\theta_x'$.

Sign convention: $\mathbf{M}_y$ multiplies $v'\,\theta_x'$ and $\mathbf{M}_z$ multiplies $w'\,\theta_x'$, both with positive sign.

The quadratic term in $\theta_x'^2$ is $(M_y\beta_z - M_z\beta_y + B\beta_w)\,\theta_x'^2$.

1. KINEMATIC RELATIONS

$$\varepsilon = \frac{du}{dx},\qquad \chi_y = \frac{d\theta_y}{dx},\qquad \chi_z = \frac{d\theta_z}{dx},\qquad \chi_\omega = \frac{d\varphi}{dx},$$

$$\gamma_y = \frac{dv}{dx} - \theta_z,\qquad \gamma_z = \frac{dw}{dx} + \theta_y,$$

$$\kappa_t = \frac{d\theta_x}{dx},\qquad \kappa_w = \frac{d\theta_x}{dx} - \varphi.$$

2. WEAK (VARIATIONAL) FORMULATION

$$\begin{aligned}
\int_0^L \Big[\, & EA\,\varepsilon\,\delta\varepsilon \\
& + EI_y\,\chi_y\,\delta\chi_y + EI_z\,\chi_z\,\delta\chi_z + EI_{yz}\,(\chi_y\,\delta\chi_z + \chi_z\,\delta\chi_y) \\
& + EI_\omega\,\chi_\omega\,\delta\chi_\omega \\
& + Gf_{tt}\,\kappa_t\,\delta\kappa_t \\
& + Gf_{yy}\,\gamma_y\,\delta\gamma_y + Gf_{zz}\,\gamma_z\,\delta\gamma_z + Gf_{ww}\,\kappa_w\,\delta\kappa_w \\
& + \frac{Gf_{yz}}{2}\,(\gamma_y\,\delta\gamma_z + \gamma_z\,\delta\gamma_y) \\
& + \frac{Gf_{yw}}{2}\,(\gamma_y\,\delta\kappa_w + \kappa_w\,\delta\gamma_y) \\
& + \frac{Gf_{zw}}{2}\,(\gamma_z\,\delta\kappa_w + \kappa_w\,\delta\gamma_z) \\
& + \frac{Gf_{tw}}{2}\,(\kappa_t\,\delta\kappa_w + \kappa_w\,\delta\kappa_t)\Big]\,dx \\
+ \int_0^L \Big[\, & N_0\,\big(v'\,\delta v' + w'\,\delta w' + z_{sc}\,(v'\,\delta\theta_x' + \theta_x'\,\delta v') \\
& \quad - y_{sc}\,(w'\,\delta\theta_x' + \theta_x'\,\delta w') + r_o^2\,\theta_x'\,\delta\theta_x'\big) \\
& + M_y\,(v'\,\delta\theta_x' + \theta_x'\,\delta v') \\
& + M_z\,(w'\,\delta\theta_x' + \theta_x'\,\delta w') \\
& + (M_y\beta_z - M_z\beta_y + B\beta_w)\,\theta_x'\,\delta\theta_x'\Big]\,dx \\
= \int_0^L & \big(q_x\,\delta u + q_y\,\delta v + q_z\,\delta w + m_x\,\delta\theta_x \\
& + m_y\,\delta\theta_y + m_z\,\delta\theta_z + b\,\delta\varphi\big)\,dx \\
+ \Big[\, & \bar{N}\,\delta u + \bar{Q}_y\,\delta v + \bar{Q}_z\,\delta w + \bar{M}_x\,\delta\theta_x \\
& + \bar{M}_y\,\delta\theta_y + \bar{M}_z\,\delta\theta_z + \bar{B}\,\delta\varphi\Big]_0^L.
\end{aligned}$$

3. STRAINS, CURVATURES AND ELASTIC (INTERNAL) STRESSES (WITHOUT GEOMETRY)

$$\begin{aligned}
N &= EA\,\frac{du}{dx}, \\
M_y &= EI_y\,\frac{d\theta_y}{dx} + EI_{yz}\,\frac{d\theta_z}{dx}, \\
M_z &= EI_z\,\frac{d\theta_z}{dx} + EI_{yz}\,\frac{d\theta_y}{dx}, \\
B &= EI_\omega\,\frac{d\varphi}{dx}, \\
Q_y^e &= Gf_{yy}\,\gamma_y + \frac{Gf_{yz}}{2}\,\gamma_z + \frac{Gf_{yw}}{2}\,\kappa_w, \\
Q_z^e &= Gf_{zz}\,\gamma_z + \frac{Gf_{yz}}{2}\,\gamma_y + \frac{Gf_{zw}}{2}\,\kappa_w, \\
M_t^e &= Gf_{tt}\,\kappa_t + \frac{Gf_{tw}}{2}\,\kappa_w, \\
T_w &= Gf_{ww}\,\kappa_w + \frac{Gf_{yw}}{2}\,\gamma_y + \frac{Gf_{zw}}{2}\,\gamma_z + \frac{Gf_{tw}}{2}\,\kappa_t.
\end{aligned}$$

4. GENERALIZED STRESSES (INCLUDING GEOMETRY)

$$\begin{aligned}
Q_y &= Q_y^e + N_0\,v' + (N_0\,z_{sc} + M_y)\,\theta_x', \\
Q_z &= Q_z^e + N_0\,w' + (-N_0\,y_{sc} + M_z)\,\theta_x', \\
M_x &= M_t^e + T_w + N_0\,r_o^2\,\theta_x' + (M_y\beta_z - M_z\beta_y + B\beta_w)\,\theta_x' \\
&\quad + (N_0\,z_{sc} + M_y)\,v' + (-N_0\,y_{sc} + M_z)\,w'.
\end{aligned}$$

5. EQUILIBRIUM EQUATIONS IN THE DOMAIN $(0, L)$

$$\begin{aligned}
dN/dx + q_x &= 0, \\
dQ_y/dx + q_y &= 0, \\
dQ_z/dx + q_z &= 0, \\
dM_x/dx + m_x &= 0, \\
dM_y/dx - Q_z + m_y &= 0, \\
dM_z/dx - Q_y + m_z &= 0, \\
dB/dx + T_w + b &= 0.
\end{aligned}$$

6. NATURAL (BOUNDARY) CONDITIONS AT $x = 0$ AND $x = L$

$N = \bar{N}$	or	$u = \bar{u}$
$Q_y = \bar{Q}_y$	or	$v = \bar{v}$
$Q_z = \bar{Q}_z$	or	$w = \bar{w}$
$M_x = \bar{M}_x$	or	$\theta_x = \bar{\theta}_x$
$M_y = \bar{M}_y$	or	$\theta_y = \bar{\theta}_y$
$M_z = \bar{M}_z$	or	$\theta_z = \bar{\theta}_z$
$B = \bar{B}$	or	$\varphi = \bar{\varphi}$

7. GEOMETRIC STIFFNESS MATRIX

The geometric stiffness matrix is obtained in closed form as

$$\mathbf{K}_{geo} = \int_0^L \mathbf{G}^T\,\mathbf{N}_0\,\mathbf{G}\,dx,$$

with $\mathbf{N}_0$ defined as

$$\mathbf{N}_0 = \begin{pmatrix}
N_0 & 0 & N_0\,z_{sc} + M_y & 0 & 0 \\
0 & N_0 & -N_0\,y_{sc} + M_z & 0 & 0 \\
N_0\,z_{sc} + M_y & -N_0\,y_{sc} + M_z & N_0\,r_o^2 + M_y\beta_z - M_z\beta_y + B\beta_w & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

DOF ORDER PER NODE (7 DOF)

1	u	(axial displacement)
2	v	(transverse y displacement)
3	w	(transverse z displacement)
4	θ_x	(torsion about x)
5	θ_y	(rotation about y)
6	θ_z	(rotation about z)
7	φ	(warping rotation)

OBSERVATION: In this formulation, the geometric effects are incorporated directly in Q_y, Q_z and M_x.
No additional terms such as $-M_z\,\theta_y' + M_y\,\theta_z'$ appear because the original couplings were replaced.
This leads to a consistent and simpler formulation for numerical implementation.