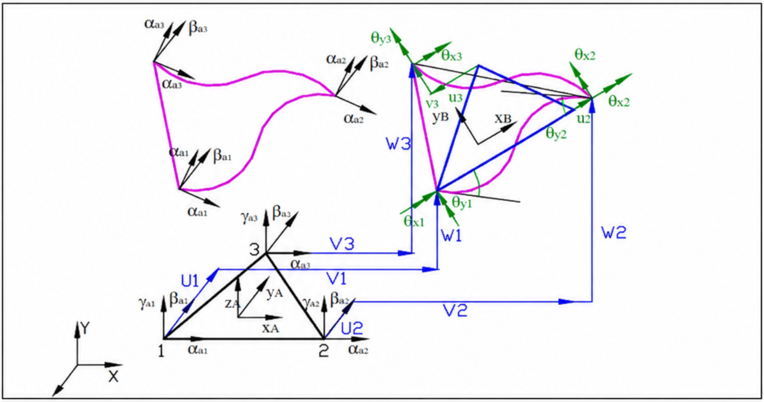
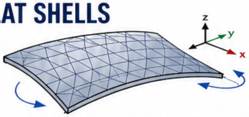


CO-ROTATIONAL FORMULATION FOR FLAT SHELLS

LARGE ROTATIONS. SMALL STRAINS, EXACT THINKING WHERE APPROXIMATIONS USUALLY SNEAK IN.

Revisiting the co-rotational framework for flat shell elements is like tuning a precision instrument: everything works—until a missing term quielly derails convergence. Below is the distilled core, sharpened and corrected for real engineering use.



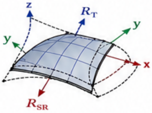
1 KINEMATIC DECOMPOSITION — MOTION WITHOUT ILLUSION

Total motion splits cleanly into rigid body rotation + pure deformation:

$$[\mathrm{R_T}] = [\mathrm{R_D}] \cdot [\mathrm{R_{SR}}]$$
$$[\mathrm{R_D}] = [\mathrm{R_T}] \cdot [\mathrm{R_{SR}}]^{\mathrm{T}}$$

In the co-rotational frame (small strains assumption):

$$[\mathrm{R_D}] \approx \mathrm{I} + [\mathrm{W_D}] = \begin{bmatrix} 1 & -\theta_z & \theta_y \\ \theta_z & 1 & -\theta_x \\ -\theta_y & \theta_x & 1 \end{bmatrix}$$

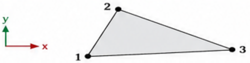


This is the quiet trick: move the complexity into rotation, keep deformation linear.

2 MINIMAL DEFORMATION SPACE — ONLY WHAT MATTERS

For a 3-node Kirchhoff shell:

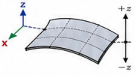
$$\{\mathrm{d_D}\} = \{\ \theta_{x1} \quad \theta_{y1} \quad u_2 \quad \theta_{x2} \quad \theta_{y2} \quad u_3 \quad v_3 \quad \theta_{x3} \quad \theta_{y3}\ \}$$



15 total DOFs – 6 rigid body modes = 9 active deformation DOFs.
No redundancy. No numerical noise.

3 STRAIN–DISPLACEMENT (LOVE–KIRCHHOFF)

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} + \varepsilon_{xr}$$
$$\varepsilon_y = \frac{\partial v}{\partial y} - z\,\frac{\partial^2 w}{\partial y^2} + \varepsilon_{yr}$$
$$\gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - 2z\,\frac{\partial^2 w}{\partial x \partial y} + \gamma_{xyr}$$



Membrane + bending coupling emerges naturally.
No shortcuts, no forgiveness.

4 ELASTIC–PLASTIC TANGENT (VON MISES, PLANE STRESS)

Isotropic linear elastic matrix [C] (plane stress):

$$[\mathrm{C}] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Flow direction:

$$\boldsymbol{n} = \frac{\partial f}{\partial \boldsymbol{\sigma}}$$

(von Mises)

$$f\,(\boldsymbol{\sigma}) = \sqrt{\frac{3}{2}\,\boldsymbol{s} : \boldsymbol{s}} - \sigma_y$$
$$\boldsymbol{s} = \boldsymbol{\sigma} - \frac{1}{3}\,\mathrm{tr}\,(\boldsymbol{\sigma})\,\mathrm{I}$$

Consistent elasto-plastic tangent:

$$[\mathrm{C_{ep}}] = [\mathrm{C}] - \frac{[\mathrm{C}] : \boldsymbol{n} \otimes \boldsymbol{n} : [\mathrm{C}]}{H' + \boldsymbol{n} : [\mathrm{C}] : \boldsymbol{n}}$$

Return mapping (radial return):

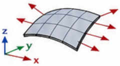
$$\boldsymbol{\sigma}_{\mathrm{C}} = \boldsymbol{\sigma}_{\mathrm{D}} - \Delta\gamma\,[\mathrm{C}] : \boldsymbol{n}$$

Solve $\Delta\gamma$ consistently—or Newton will punish you.

H' = algorithmic hardening modulus (e.g., $H' = d\sigma_y / d\bar{\varepsilon}_p$ for istvtropic hardening)

5 INTERNAL FORCE TRANSFORMATION

$$\{f_{\mathrm{int}}\}^{\mathrm{A}} = [\mathrm{R_{SR}}] \cdot [\mathrm{H}] \cdot \{f_{\mathrm{int}}\}_{\mathrm{D}}$$



A simple mapping, but fully nonlinear in disguise.

6 THE CRITICAL POINT — TANGENT STIFFNESS IS NOT 2 TERMS

$$[\mathrm{K_T}] = \frac{d[\mathrm{R_{SR}}]}{d\{\mathrm{d_T}\}}\ [\mathrm{H}]\ \{f_{\mathrm{fint}}\}_{\mathrm{D}}$$ ← geometric (stress stiffness)

$$+\ [\mathrm{R_{SR}}]\ \frac{d[\mathrm{H}]}{d\{\mathrm{d_T}\}}\ \{f_{\mathrm{fint}}\}_{\mathrm{D}}$$ ← mapping variation (often ignored)

$$+\ [\mathrm{R_{SR}}]\ [\mathrm{H}]\ [\mathrm{K_{mat}}]\ [\mathrm{B_{geo}}]$$ ← material + deformation coupling

THREE TERMS. ALWAYS THREE.

Omit the second, and you're not simplifying—you're breaking consistency.
That "negligible" term becomes decisive when:

- shape functions are configuration-dependent
- drilling DOFs are present
- distorted elements enter the game

And suddenly: no quadratic convergence, residuals that refuse to die, and hours lost chasing ghosts.

7 WHY THIS FORMULATION STILL DOMINATES

- Clean separation: rigid motion vs deformation
- Exact handling of large rotations
- Compatible with nonlinear material laws
- Restores quadratic convergence when fully consistent

This is why it sits at the heart of industrial solvers.

YOUR TURN

Have you seen convergence issues traced back to an incomplete tangent?
Co-rotational or Total Lagrangian—what's your battlefield experience?

The co-rotational method is not just elegant—
it's unforgiving.
It rewards rigor and exposes shortcuts.
Ignore a term, and the algorithm whispers at first...