THE MATRIX STIFFNESS METHOD

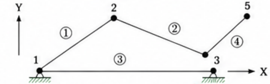
(BASED ON CHAPTER 7 – MATRIX ANALYSIS)

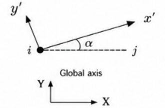
The Matrix Stiffness Method is a systematic procedure for analyzing structures by determining the nodal displacements and member forces using matrix algebra.

- i → node number
- m → element number
- m → number of nodes
- m_{dof} → number of degrees of freedom (DOF)
- u → number of unrestrained DOF ($u = m_{dof} - r$)

- ● node
- — element
- support (restraint)
- Applied load (known)
- R_x, R_y, M support reactions (unknowns)

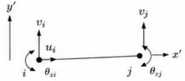
1. GLOBAL AND LOCAL AXES





2. RELATIONSHIP BETWEEN FORCES AND DISPLACEMENTS IN LOCAL AXIS

Element "m" (i–j) in local axis



Nodal displacement vector in local axis (element m):

$$\{d^m\} = \{u_i \quad v_i \quad \theta_{zi} \quad u_j \quad v_j \quad \theta_{zj}\}^T$$

Stiffness matrix in local axis (2D frame: axial + bending)

$$[k^m] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

where L = element length

E = modulus of elasticity A = cross-sectional area I = moment of inertia (about z)

Assumptions:
1- Euler-Bernoulli: Neglects deformation due to shear forces
2- Timoshenko: Takes into account deformation due to shear forces

If shear deformation is considered:

3. TRANSFORMATION TO GLOBAL AXIS

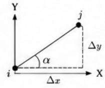
$$\{d_g^m\} = [T]^T \{d^m\} \quad ; \quad \{f_g^m\} = [T]^T \{f^m\}$$

$$[k_g^m] = [T]^T [k^m] [T]$$

Transformation matrix $[T]$ (2D frame)

$c = \cos\alpha \qquad s = \sin\alpha$

$$[T] = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

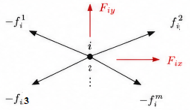


$$\alpha = \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right)$$

Displacements, forces and stiffness in global axes are obtained with the transformation matrix $[T]$.

4. EQUILIBRIUM EQUATIONS AT THE NODES

At node i (example):



Equilibrium at node i

External (known) nodal loads:

$$\{F^{ext}\} = \begin{Bmatrix} F_{ix} \\ F_{iy} \\ M_i \end{Bmatrix}$$

Internal (from connected elements):

$$\{F^{int}\} = -\sum_{m \in i j} \{f^m\}$$

Where $\{f^m\}$ are the end forces at node i of element m (in global axes).

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0$$

5. EQUILIBRIUM EQUATIONS OF THE STRUCTURE

$\sum F_x = 0$
$\sum F_y = 0$
$\sum M_z = 0$

In matrix form:

$$[K]\ \{d\} = \{F_{ext}\}$$

Building global stiffness matrix $[K]$:

For each element $m = i$–j in global axes

- Add the 6x6 matrix $[k_g^m]$ to $[K]$ in the rows and columns corresponding to the DOF of nodes i and j.
- No force component is assumed zero.

Where:
$\{d\}$ = global nodal displacement vector (containing u, v, θ_z at each node)
$\{F_{ext}\}$ = global nodal load vector (containing F_x, F_y, M_z at each node)

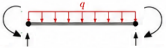
Global DOF at each node (2D frame):

$$\{d_i\} = \{u_i \quad v_i \quad \theta_{zi}\}^T$$

Total DOF $= 3n$

6. LOAD BETWEEN NODES

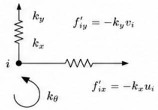
Divide the analysis into two states.
State 1: Apply forces and moments at the nodes to restrain the movements at the nodes.



State 2: Apply opposite forces and moments at the nodes to the ones applied in the previous state.

Add the results of both states.

7. SUPPORTS WITH SPRINGS



$f'_{iy} = -k_y v_i$

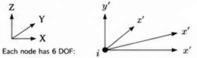
$f'_{ix} = -k_x u_i$

$m'_i = -k_\theta \theta_{zi}$

$$\begin{Bmatrix} F_x \\ F_y \\ M_z \\ \vdots \end{Bmatrix} = \begin{bmatrix} K_{uu} & K_{ur} \\ K_{ru} & K_{rr} \end{bmatrix} \begin{Bmatrix} d_u \\ d_r \end{Bmatrix} + \begin{Bmatrix} F_x^{spr} \\ F_y^{spr} \\ M^{spr} \\ \vdots \end{Bmatrix}$$

with
d_u = unrestrained displacements
d_r = restrained displacements (usually = 0)
F^{spr} = spring force vector (acts at restrained DOF)

8. 3D BEHAVIOR (SPACE FRAME)



Each node has 6 DOF:
$u, v, w, \theta_x, \theta_y, \theta_z$
(3 translations + 3 rotations)

$[k^m]$ = 12 × 12 stiffness matrix (axial + torsion + bending about two axes)

9. GEOMETRIC NONLINEAR ANALYSIS (BUCKLING LOAD)

Buckling load: Obtain the load factor λ that cancels the determinant of the stiffness matrix of the structure.

$$det\ ([K] + \lambda [K_G]) = 0$$

Where $[K_G]$ = geometric stiffness matrix.

10. SUMMARY OF PROCEDURE

1. Idealize the structure →
2. Assign DOF (global axes) →
3. Apply supports and loads (build $\{F_{ext}\}$) →
4. Obtain element stiffness matrices (local axes) →
5. Transform to global axes and assemble $[K]$ →
6. Apply boundary conditions and solve for $\{d_u\}$ →
7. Compute reactions $\{F_r\}$ →
8. Compute member forces

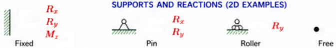
NOTATION

- $\{d\}$: nodal displacement vector
- $\{F\}$: nodal force vector (F_x, F_y, M_z)
- $[K]$: global stiffness matrix
- $[k^m]$: element stiffness matrix
- $[T]$: transformation matrix

DISPLACEMENTS / DOF AT A NODE (2D FRAME)

u (x-displacement) $\quad v$ (y-displacement) $\quad \theta_z$ (rotation about z)

SUPPORTS AND REACTIONS (2D EXAMPLES)



ELEMENT NUMBERING

$m = 1, 2, \ldots, m_{el}$

$i \longrightarrow j$

$i < j$