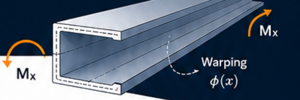
EXACT vs. LINEAR FINITE ELEMENT FORMULATION IN TORSION WITH WARPING

When One Element Beats a Mesh



Torsion of thin-walled open sections is governed by the coupling between torsional rotation and warping.
A key parameter controls this behavior:

$$k = 1 - \frac{J}{I_p}$$

J = Saint-Venant torsional constant
I_p = Polar moment of inertia

For open sections: $J << I_p \Rightarrow k \approx 1$ (strong warping)

For closed sections: $J \to I_p \Rightarrow k \to 0$ (classical torsion)

1) THE MECHANICAL VARIABLES

Generalized displacements

$$d = [\,\theta_x\,,\ \phi\,]^T$$

θ_x = torsional rotation
ϕ = warping deformation

Generalized forces

$$f = [\,M_x\,,\ B_\omega\,]^T$$

M_x = torsional moment
B_ω = bimoment

Element length: L | Torsional stiffness: GJ | Warping stiffness: EI_ω

2) EXACT FORMULATION (TRANSFER MATRIX)

The governing equations can be written as a first-order system:

$$\frac{d}{dx}\begin{bmatrix}\theta_x\\ \phi\\ M_x\\ B_\omega\end{bmatrix} = A\begin{bmatrix}\theta_x\\ \phi\\ M_x\\ B_\omega\end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix}0 & k & (1-k)/GJ & 0\\ 0 & 0 & 0 & 1/EI_\omega\\ 0 & 0 & 0 & 0\\ 0 & k\cdot GJ & -k & 0\end{bmatrix}$$

The solution is exact and expressed through the transfer matrix:

$$E(L) = T\cdot E(0), \quad \text{where} \quad T = \exp(A\cdot L)$$

From the blocks of T, the exact stiffness matrix is obtained:

$$K_{11} = -inv(T_{12})\cdot T_{11} \qquad K_{21} = -T_{21} + T_{22}\cdot inv(T_{12})\cdot T_{11}$$

$$K_{12} = inv(T_{12}) \qquad K_{22} = -T_{22}\cdot inv(T_{12})$$

$$K_{\text{exact}} = \begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}$$

- Analytical solution
- One element is enough
- No discretization error

Key idea:
The structure is not approximated — it is **propagated.**

3) LINEAR FINITE ELEMENT FORMULATION

Weak form:

$$\int_0^L \left[EI_\omega\,\phi'\,\delta\phi' + \frac{kGJ}{1-k}\,(\theta_x' - \phi)(\delta\theta_x' - \delta\phi) + GJ\ \theta_x'\,\delta\theta_x'\right] dx$$

$$= \int_0^L (m_x\,\delta\theta_x + b_\omega\,\delta\phi)\,dx + \Big[M_x\,\delta\theta_x + B_\omega\,\delta\phi\Big]_0^L$$

Linear interpolation:

$$\theta_x(x) = N_1(x)\,\theta_{x1} + N_2(x)\,\theta_{x2}$$

$$\phi(x) = N_1(x)\,\phi_1 + N_2(x)\,\phi_2$$

$$N_1 = 1 - x/L \qquad N_2 = x/L$$

Resulting stiffness matrix (order: θ_{x1}, ϕ_1, θ_{x2}, ϕ_2):

$$K_{FEM} = \begin{bmatrix}
\frac{GJ}{(1-k)L} & \frac{kGJ}{2(1-k)} & \frac{-GJ}{(1-k)L} & \frac{kGJ}{2(1-k)}\\
\frac{kGJ}{2(1-k)} & \frac{EI_\omega}{L} + \frac{kGJ\cdot L}{3(1-k)} & \frac{-kGJ}{2(1-k)} & \frac{EI_\omega}{L} + \frac{kGJ\cdot L}{6(1-k)}\\
\frac{-GJ}{(1-k)L} & \frac{-kGJ}{2(1-k)} & \frac{GJ}{(1-k)L} & \frac{-kGJ}{2(1-k)}\\
\frac{kGJ}{2(1-k)} & \frac{EI_\omega}{L} + \frac{kGJ\cdot L}{6(1-k)} & \frac{-kGJ}{2(1-k)} & \frac{EI_\omega}{L} + \frac{kGJ\cdot L}{3(1-k)}
\end{bmatrix}$$

Simple to implement
Easy to assemble

Approximation error
Requires many elements

4) VLASOV CASE: EQUAL WARPING AND ROTATION DERIVATIVE (Applicable to open sections when $k \to 1$)

In Vlasov theory for open thin-walled sections under restrained warping, the minimization of strain energy leads to:

$$\phi(x) = \theta_x'(x) \quad \text{(Vlasov condition)}$$

This kinematic constraint implies that warping is equal to the derivative of torsional rotation along the member.

Applicable when: $k \to 1$
(open sections, strong warping)

To represent this behavior accurately, finite elements must use cubic polynomial shape functions (Hermitian-type).

Cubic shape functions (example)

$$N_1(\xi) = 1 - 3\xi^2 + 2\xi^3$$

$$N_2(\xi) = L(\xi - 2\xi^2 + \xi^3)$$

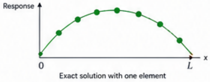
$$N_3(\xi) = 3\xi^2 - 2\xi^3$$

$$N_4(\xi) = L(-\xi^2 + \xi^3)$$

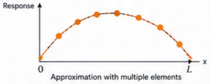
$$\xi = \frac{x}{L} \in [0,1]$$

5) WHAT'S THE REAL DIFFERENCE?

Exact formulation
Captures the full coupling between torsion and warping for any length L.
No mesh. No iteration.
Just the exact physics.



Linear FEM
Approximates the fields with linear interpolation.
Accuracy improves only as the mesh is refined.



6) ENGINEERING JUDGMENT

Use the exact formulation when:

- You need benchmark or validation solutions
- You want physical insight
- The member is uniform (constant properties)

Use linear FEM when:

- Geometry is complex
- Properties vary along the member
- The structure is large-scale and general

**Discretization is a powerful tool.
But sometimes, the exact solution is just one exponential away.**

☆ *In engineering, elegance is not in the equation — it is in the decision.*